

# Cluster $C^*$ -algebras and knot polynomials

Igor Nikolaev

## Abstract

We construct a representation of the braid groups in a cluster  $C^*$ -algebra coming from a triangulation of the Riemann surface  $S$  with one or two cusps. It is shown that the Laurent polynomials attached to the  $K$ -theory of such an algebra are topological invariants of the closure of braids. In particular, the Jones and HOMFLY polynomials of a knot correspond to the case  $S$  being a sphere with two cusps and a torus with one cusp, respectively.

*Key words and phrases:* cluster  $C^*$ -algebras, Jones polynomials

*MSC:* 13F60 (cluster algebras); 46L85 (noncommutative topology); 57M25 (knots and links)

## 1 Introduction

The *trace invariant*  $V_L(t)$  of a link  $L$  was introduced in [Jones 1985] [10]. The  $V_L(t)$  is a Laurent polynomial  $\mathbf{Z}[t^{\pm\frac{1}{2}}]$  obtained from a representation of the braid group  $B_k$  in an operator (von Neumann) algebra  $A_k$ ; the canonical trace on  $A_k$  times a multiple  $\frac{t}{(1+t)^2}$  is invariant of the Markov move of type II of a braid  $b \in B_k$ . The Jones polynomial  $V_L(t)$  is a powerful topological invariant of a link  $L$  obtained by the closure of  $b$ . The algebra  $A_k$  itself comes from an analog of the Galois theory for von Neumann algebras called *basic construction* [Jones 1991, Section 2.6] [11]. It is yet unclear why the braid groups appear in the context of operator algebras, let alone an invariant trace with such a remarkable property; likewise, one can seek to extend the invariant  $V_L(t)$  to the multivariable Laurent polynomials.

*Cluster algebras* are a class of commutative rings introduced by [Fomin & Zelevinsky 2002] [7] having deep roots in hyperbolic geometry and Teichmüller theory [Williams 2014] [16]. Such an algebra  $\mathcal{A}(\mathbf{x}, B)$  is a sub-ring of the field of rational functions in  $n$  variables depending on a cluster  $\mathbf{x} = (x_1, \dots, x_n; y_1, \dots, y_m)$  of *mutable* variables  $x_i$  and *frozen* variables  $y_i$  and a skew-symmetric matrix  $B = (b_{ij}) \in M_n(\mathbf{Z})$ ; the pair  $(\mathbf{x}, B)$  is called a *seed*. In terms of the coefficients  $c_i$  from a semi-field  $(\mathbb{P}, \oplus, \bullet)$  a new cluster  $\mathbf{x}' = (x_1, \dots, x'_k, \dots, x_n; c_1, \dots, c'_j, \dots, c_n)$  and a new skew-symmetric matrix  $B' = (b'_{ij})$  is obtained from  $(\mathbf{x}, B)$  by the exchange relations:

$$\begin{aligned} b'_{ij} &= \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise,} \end{cases} \\ c'_j &= \begin{cases} \frac{1}{c_k} & \text{if } j = k \\ \frac{c_j c_k^{\max(b_{kj}, 0)}}{(c_k \oplus 1)^{b_{kj}}} & \text{otherwise,} \end{cases} \\ x'_k &= \frac{c_k \prod_{i=1}^n x_i^{\max(b_{ik}, 0)} + \prod_{i=1}^n x_i^{\max(-b_{ik}, 0)}}{(c_k \oplus 1) x_k}, \end{aligned} \quad (1)$$

see [Williams 2014, Definition 2.22] [16] for the details. The seed  $(\mathbf{x}', B')$  is said to be a *mutation* of  $(\mathbf{x}, B)$  in direction  $k$ , where  $1 \leq k \leq n$ ; the algebra  $\mathcal{A}(\mathbf{x}, B)$  is generated by cluster variables  $\{x_i\}_{i=1}^\infty$  obtained from the initial seed  $(\mathbf{x}, B)$  by the iteration of mutations in all possible directions  $k$ . The *Laurent phenomenon* proved by [Fomin & Zelevinsky 2002] [7] says that  $\mathcal{A}(\mathbf{x}, B) \subset \mathbf{Z}[\mathbf{x}^{\pm 1}]$ , where  $\mathbf{Z}[\mathbf{x}^{\pm 1}]$  is the ring of the Laurent polynomials in variables  $\mathbf{x} = (x_1, \dots, x_n)$ ; in other words, each generator  $x_i$  of algebra  $\mathcal{A}(\mathbf{x}, B)$  can be written as a Laurent polynomial in  $n$  variables with the integer coefficients. (Note that the Laurent phenomenon turns  $\mathcal{A}(\mathbf{x}, B)$  into an additive abelian group with an order coming from the semi-group of the Laurent polynomials with positive coefficients.) In what follows, we deal with a cluster algebra  $\mathcal{A}(\mathbf{x}, S_{g,n})$  coming from a triangulation of the Riemann surface  $S_{g,n}$  of genus  $g$  with  $n$  cusps, see [Fomin, Shapiro & Thurston 2008] [8] the details.

*Cluster  $C^*$ -algebras* are a class of non-commutative rings  $\mathbb{A}(\mathbf{x}, B)$ , such that  $K_0(\mathbb{A}(\mathbf{x}, B)) \cong \mathcal{A}(\mathbf{x}, B)$  [14]; here  $K_0(\mathbb{A}(\mathbf{x}, B))$  is the  $K_0$ -group of a  $C^*$ -algebra  $\mathbb{A}(\mathbf{x}, B)$  and  $\cong$  is an isomorphism of the additive abelian groups with order [Blackadar 1986] [2]. The  $\mathbb{A}(\mathbf{x}, B)$  is an *AF-algebra* given by the Bratteli diagram [Bratteli 1972] [3]; such a diagram can be obtained from a mutation tree of the initial seed  $(\mathbf{x}, B)$  modulo an equivalence relation be-

tween the seeds lying at the same level. It is known that the  $AF$ -algebras are characterized by their  $K$ -theory [Elliott 1976] [4]. Equivalently, the  $\mathbb{A}(\mathbf{x}, B)$  is an algebra over the complex numbers generated by a series of projections  $\{e_i\}_{i=1}^\infty$ .

The aim of our note is a representation of the braid group  $B_k = \{\sigma_1, \dots, \sigma_{k-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2\}$  into an algebra  $\mathbb{A}(\mathbf{x}, S_{g,n})$ , so that the Laurent phenomenon in  $K_0(\mathbb{A}(\mathbf{x}, S_{g,n}))$  corresponds to the polynomial invariants of the closure of braids  $b \in B_k$ . In particular, if  $g = 0$  and  $n = 2$  or  $g = n = 1$  one recovers the Jones invariant  $V_L(t)$  or the HOMFLY polynomials of knots [Freyd, Yetter, Hoste, Lickorish, Millet & Ocneanu 1985] [9], respectively. Whenever  $g \geq 2$  one gets new topological invariants generalizing the Jones and HOMFLY polynomials to an arbitrary (but finite) number of variables. The  $AF$ -algebra  $\mathbb{A}(\mathbf{x}, S_{g,n})$  itself can be viewed as an analog of the tower  $\cup_{k=1}^\infty A_k$  of von Neumann algebras  $A_k$  arising from the basic construction [Jones 1991, Section 3.4] [11]. Unlike [Jones 1985] [10], we exploit the phenomenon of cluster algebras and the *Birman-Hilden Theorem* relating the braid groups  $B_{2g+n}$  with the mapping class group of surface  $\{S_{g,n} \mid n = 1; 2\}$  [Birman & Hilden 1971] [1]. Our main result can be formulated as follows.

**Theorem 1** *The formula  $\sigma_i \mapsto e_i + 1$  defines a representation*

$$\rho : \begin{cases} B_{2g+1} \rightarrow \mathbb{A}(\mathbf{x}, S_{g,1}) \\ B_{2g+2} \rightarrow \mathbb{A}(\mathbf{x}, S_{g,2}). \end{cases} \quad (2)$$

*If  $b \in B_{2g+1}$  ( $b \in B_{2g+2}$ , resp.) is a braid, there exists a Laurent polynomial  $[\rho(b)] \in K_0(\mathbb{A}(\mathbf{x}, S_{g,1}))$  ( $[\rho(b)] \in K_0(\mathbb{A}(\mathbf{x}, S_{g,2}))$ , resp.) with the integer coefficients depending on  $2g$  ( $2g + 1$ , resp.) variables, such that  $[\rho(b)]$  is a topological invariant of the closure of  $b$ .*

The article is organized as follows. We introduce preliminary facts and notation in Section 2. Theorem 1 is proved in Section 3. To illustrate theorem 1 in Section 4, we consider the cases  $g = 0$ ,  $n = 2$  and  $g = n = 1$  corresponding to the Jones and HOMFLY polynomials, respectively.

## 2 Preliminaries

We shall briefly review the Birman-Hilden Theorem, the cluster  $C^*$ -algebras and knots. We refer the reader to [Birman & Hilden 1971] [1], [Blackadar

1986] [2], [Farb & Margalit 2011] [5], [Fomin, Shapiro & Thurston 2008] [8], [Jones 1985] [10], [Jones 1991, Lecture 6] [11], [Williams 2014] [16], [13] and [14] for a detailed account.

## 2.1 Birman-Hilden Theorem

Let  $S_{g,n}$  be a Riemann surface of genus  $g \geq 0$  with  $n \geq 1$  cusps and such that  $2g - 2 + n > 0$ ; denote by  $T_{g,n} \cong \mathbb{R}^{6g-6+2n}$  the (decorated) Teichmüller space of  $S_{g,n}$ , i.e. a collection of all Riemann surfaces of genus  $g$  with  $n$  cusps endowed with the natural topology. By  $\text{Mod } S_{g,n}$  we understand the mapping class group of surface  $S_{g,n}$ , i.e. a group of the homotopy classes of orientation preserving automorphisms of  $S_{g,n}$  fixing all cusps. It is well known that two Riemann surfaces  $S, S' \in T_{g,n}$  are isomorphic if and only if there exists a  $\varphi \in \text{Mod } S_{g,n}$ , such that  $S' = \varphi(S)$ ; thus each  $\varphi \in \text{Mod } S_{g,n}$  corresponds to a homeomorphism of the Teichmüller space  $T_{g,n}$ . If  $\gamma$  is a simple closed curve on  $S_{g,n}$ , let  $D_\gamma \in \text{Mod } S_{g,n}$  be the Dehn twist around  $\gamma$ ; a pair of the Dehn twists  $D_{\gamma_i}$  and  $D_{\gamma_j}$  satisfy the braid relations:

$$\begin{cases} D_{\gamma_i} D_{\gamma_j} D_{\gamma_i} = D_{\gamma_j} D_{\gamma_i} D_{\gamma_j}, & \text{if } \gamma_i \cap \gamma_j = \{\text{single point}\} \\ D_{\gamma_i} D_{\gamma_j} = D_{\gamma_j} D_{\gamma_i}, & \text{if } \gamma_i \cap \gamma_j = \emptyset. \end{cases} \quad (3)$$

A system of simple closed curves  $\{\gamma_i\}$  on  $S_{g,n}$  is called a *chain*, if

$$\begin{cases} \gamma_i \cap \gamma_{i+1} = \{\text{single point}\} \\ \gamma_i \cap \gamma_j = \emptyset & \text{otherwise.} \end{cases} \quad (4)$$

Consider a chain  $\{\gamma_1, \dots, \gamma_{2g+1}\}$  shown in [Farb & Margalit 2011, Figure 2.7] [5]. The fundamental domain of  $S_{g,n}$  obtained by a cut along the chain is a  $(4g+2)$ -gon with the opposite sides identified [Farb & Margalit 2011, Figure 2.2] [5]. A *hyperelliptic involution*  $\iota \in \text{Mod } S_{g,n}$  is a rotation by the angle  $\pi$  of the  $(4g+2)$ -gon; clearly, the chain  $\{\gamma_1, \dots, \gamma_{2g+1}\}$  is an invariant of the involution  $\iota$ .

In what follows, we focus on the case  $n = 1$  or  $2$ . Notice that  $S_{g,1}$  ( $S_{g,2}$ , resp.) can be replaced by a surface of the same genus with one (two, resp.) boundary components and no cusps; since the mapping class group preserves the boundary components, we work with this new surface while keeping the old notation. It is known that  $S_{g,1}$  ( $S_{g,2}$ , resp.) is a double cover of a disk  $\mathcal{D}_{2g+1}$  ( $\mathcal{D}_{2g+2}$ , resp.) ramified at the  $2g+1$  ( $2g+2$ , resp.) inner points of the disk; we refer the reader to [Farb & Margalit 2011, Figure 9.15] [5]

for a picture. It transpires that each automorphism of  $\mathcal{D}_{2g+1}$  ( $\mathcal{D}_{2g+2}$ , resp.) pulls back to an automorphism of  $S_{g,1}$  ( $S_{g,2}$ , resp.) commuting with the hyperelliptic involution  $\iota$ . Recall that  $\text{Mod } \mathcal{D}_{2g+1} \cong B_{2g+1}$  ( $\text{Mod } \mathcal{D}_{2g+2} \cong B_{2g+2}$ , resp.); a subgroup of  $\text{Mod } S_{g,1}$  ( $\text{Mod } S_{g,2}$ , resp.) commuting with  $\iota$  is called *symmetric* and denoted by  $\text{SMod } S_{g,1}$  ( $\text{SMod } S_{g,2}$ , resp.) The following result is critical.

**Theorem 2** ([Birman & Hilden 1971] [1]) *There exists an isomorphism:*

$$\begin{cases} B_{2g+1} \cong \text{SMod } S_{g,1} \\ B_{2g+2} \cong \text{SMod } S_{g,2} \end{cases} \quad (5)$$

given by the formula  $\sigma_i \mapsto D_{\gamma_i}$ , where  $\sigma_i$  is a generator of the braid group  $B_{2g+1}$  ( $B_{2g+2}$ , resp.) and  $D_{\gamma_i}$  is the Dehn twist around the simple closed curve  $\gamma_i$  of a chain in the  $S_{g,1}$  ( $S_{g,2}$ , resp.)

## 2.2 Cluster $C^*$ -algebras from Riemann surfaces

The fundamental domain of the Riemann surface  $S_{g,n}$  has a triangulation by the  $6g - 6 + 3n$  geodesic arcs  $\gamma_i$ ; the endpoint of each  $\gamma_i$  is a cusp at the absolute of Lobachevsky plane  $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$ . Denote by  $l(\gamma_i)$  the  $\pm$  hyperbolic length of  $\gamma_i$  between two horocycles around the endpoints of  $\gamma_i$ ; consider the  $\lambda(\gamma_i) = \exp(\frac{1}{2}l(\gamma_i))$ . The following result says that  $\lambda(\gamma_i)$  are coordinates in the Teichmüller space  $T_{g,n}$ .

**Theorem 3** ([Penner 1987] [15]) *The map  $\lambda : \{\gamma_i\}_{i=1}^{6g-6+3n} \rightarrow T_{g,n}$  is a homeomorphism.*

**Remark 1** The six-tuples of numbers  $\lambda(\gamma_i)$  must satisfy the *Ptolemy relation*  $\lambda(\gamma_1)\lambda(\gamma_2) + \lambda(\gamma_3)\lambda(\gamma_4) = \lambda(\gamma_5)\lambda(\gamma_6)$ , where  $\gamma_1, \dots, \gamma_4$  are pairwise opposite sides and  $\gamma_5, \gamma_6$  are the diagonals of a geodesic quadrilateral in  $\mathbb{H}$ . The  $n$  Ptolemy relations reduce the number of independent variables  $\lambda(\gamma_i)$  to  $6g - 6 + 2n = \dim T_{g,n}$ .

Let  $T = \{\gamma_i\}_{i=1}^{6g-6+3n}$  be a triangulation of  $S_{g,n}$ ; consider a skew-symmetric matrix  $B_T = (b_{ij})$  of rank  $6g - 6 + 3n$ , where  $b_{ij}$  is equal to the number of triangles in  $T$  with sides  $\gamma_i$  and  $\gamma_j$  in clockwise order minus the number of triangles in  $T$  with sides  $\gamma_i$  and  $\gamma_j$  in the counter-clockwise order. (For a quick example of matrix  $B_T$  we refer the reader to Section 4.)

**Theorem 4** ([Fomin, Shapiro & D. Thurston 2008] [8]) *The cluster algebra  $\mathcal{A}(\mathbf{x}, B_T)$  does not depend on triangulation  $T$ , but only on the surface  $S_{g,n}$ ; namely, replacement of the geodesic arc  $\gamma_k$  by a new geodesic arc  $\gamma'_k$  (a flip of  $\gamma_k$ ) corresponds to a mutation  $\mu_k$  of the seed  $(\mathbf{x}, B_T)$ .*

**Definition 1** *The algebra  $\mathcal{A}(\mathbf{x}, S_{g,n}) := \mathcal{A}(\mathbf{x}, B_T)$  is called a cluster algebra of the Riemann surface  $S_{g,n}$ .*

A  $C^*$ -algebra is an algebra  $A$  over  $\mathbf{C}$  with a norm  $a \mapsto \|a\|$  and an involution  $a \mapsto a^*$  such that it is complete with respect to the norm and  $\|ab\| \leq \|a\| \|b\|$  and  $\|a^*a\| = \|a\|^2$  for all  $a, b \in A$ . Any commutative  $C^*$ -algebra is isomorphic to the algebra  $C_0(X)$  of continuous complex-valued functions on some locally compact Hausdorff space  $X$ ; otherwise,  $A$  represents a noncommutative topological space. For a unital  $C^*$ -algebra  $A$ , let  $V(A)$  be the union (over  $n$ ) of projections in the  $n \times n$  matrix  $C^*$ -algebra with entries in  $A$ ; projections  $p, q \in V(A)$  are (Murray - von Neumann) *equivalent* if there exists a partial isometry  $u$  such that  $p = u^*u$  and  $q = uu^*$ . The equivalence class of projection  $p$  is denoted by  $[p]$ ; the equivalence classes of orthogonal projections can be made to a semigroup by putting  $[p] + [q] = [p + q]$ . The Grothendieck completion of this semigroup to an abelian group is called the  $K_0$ -group of the algebra  $A$ . The functor  $A \rightarrow K_0(A)$  maps the category of unital  $C^*$ -algebras into the category of abelian groups, so that projections in the algebra  $A$  correspond to a positive cone  $K_0^+ \subset K_0(A)$  and the unit element  $1 \in A$  corresponds to an order unit  $u \in K_0(A)$ . The ordered abelian group  $(K_0, K_0^+, u)$  with an order unit is called a *dimension group*; an order-isomorphism class of the latter we denote by  $(G, G^+)$ .

An *AF-algebra*  $\mathbb{A}$  (Approximately Finite  $C^*$ -algebra) is defined to be the norm closure of an ascending sequence of finite dimensional  $C^*$ -algebras  $M_n$ , where  $M_n$  is the  $C^*$ -algebra of the  $n \times n$  matrices with entries in  $\mathbf{C}$ ; such an algebra is given by an infinite graph called *Bratteli diagram*, see [Bratteli 1972] [3] for a definition. The dimension group  $(K_0(\mathbb{A}), K_0^+(\mathbb{A}), u)$  is a complete isomorphism invariant of the algebra  $\mathbb{A}$  [Elliott 1976] [4]. The order-isomorphism class  $(K_0(\mathbb{A}), K_0^+(\mathbb{A}))$  is an invariant of the *Morita equivalence* of algebra  $\mathbb{A}$ , i.e. an isomorphism class in the category of finitely generated projective modules over  $\mathbb{A}$ . The *scale*  $\Gamma$  is a subset of  $K_0^+(\mathbb{A})$  which is generating, hereditary and directed, i.e. (i) for each  $a \in K_0^+(\mathbb{A})$  there exist  $a_1, \dots, a_r \in \Gamma(\mathbb{A})$ , such that  $a = a_1 + \dots + a_r$ ; (ii) if  $0 \leq a \leq b \in \Gamma$ , then  $a \in \Gamma$ ; (iii) given  $a, b \in \Gamma$  there exists  $c \in \Gamma$ , such that  $a, b \leq c$ . If  $u$  is an

order unit, then the set  $\Gamma := \{a \in K_0^+(\mathbb{A}) \mid 0 \leq a \leq u\}$  is a scale; thus the dimension group of algebra  $\mathbb{A}$  can be written in the form  $(K_0(\mathbb{A}), K_0^+(\mathbb{A}), \Gamma)$ .

**Definition 2** ([14]) *By a cluster  $C^*$ -algebra  $\mathbb{A}(\mathbf{x}, S_{g,n})$  one understands an AF-algebra satisfying an isomorphism of the scaled dimension groups:*

$$(K_0(\mathbb{A}(\mathbf{x}, S_{g,n})), K_0^+(\mathbb{A}(\mathbf{x}, S_{g,n})), u) \cong (\mathcal{A}(\mathbf{x}, S_{g,n}), \mathcal{A}^+(\mathbf{x}, S_{g,n}), u'), \quad (6)$$

where  $\mathcal{A}^+(\mathbf{x}, S_{g,n})$  is a semi-group of the Laurent polynomials with positive coefficients and  $u'$  is an order unit in  $\mathcal{A}^+(\mathbf{x}, S_{g,n})$ .

**Remark 2** Theorems 3 and 4 imply that the algebra  $\mathbb{A}(\mathbf{x}, S_{g,n})$  is a non-commutative coordinate ring of the Teichmüller space  $T_{g,n}$ ; in other words, the diagram in Figure 1 must be commutative.

$$\begin{array}{ccc} & \text{homeomorphism} & \\ T_{g,n} & \xrightarrow{\quad} & T_{g,n} \\ \downarrow & & \downarrow \\ \mathbb{A}(\mathbf{x}, S_{g,n}) & \xrightarrow[\text{automorphism}]{\text{inner}} & \mathbb{A}(\mathbf{x}, S_{g,n}) \end{array}$$

Figure 1: Coordinate ring of the space  $T_{g,n}$ .

**Remark 3** The  $\mathbb{A}(\mathbf{x}, S_{g,n})$  is the norm-closure of an algebra of the non-commutative polynomials  $\mathbf{C}\langle e_1, e_2, \dots \rangle$ , where  $\{e_i\}_{i=1}^\infty$  are projections in the algebra  $\mathbb{A}(\mathbf{x}, S_{g,n})$ ; this fact follows from the  $K$ -theory of  $\mathbb{A}(\mathbf{x}, S_{g,n})$ . On the other hand, the algebra  $\mathcal{A}(\mathbf{x}, S_{g,n})$  is generated by the cluster variables  $\{x_i\}_{i=1}^\infty$ . We shall denote by  $\rho(x_i) = e_i$  a natural bijection between the two sets of generators.

## 2.3 Knots and links

A *knot* is a tame embedding of the circle  $S^1$  into the Euclidean space  $\mathbf{R}^3$ ; a *link* with the  $n$  components is such an embedding of the union  $S^1 \cup \dots \cup S^1$ .

The classification of (distinct) knots and links is a difficult open problem of topology. The *Alexander Theorem* says that every knot or link comes from the closure of a braid  $b \in B_k = \{\sigma_1, \dots, \sigma_{k-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2\}$ , i.e. tying the top end of each string of  $b$  to the end of a string in the same position at the bottom. Thus the braids can classify knots and links but sadly rather “unrelated” braids  $b \in B_k$  and  $b' \in B_{k'}$  can produce the same knot. Namely, the *Markov Theorem* says that the closure of braids  $b, b' \in B_k$  corresponds to the same knot or link, if and only if: (i)  $b' = gbg^{-1}$  for a braid  $g \in B_k$  or (ii)  $b' = b\sigma_k^{\pm 1}$  for the generator  $\sigma_k \in B_{k+1}$ . Thus the *Markov move of type II* always pushes “to infinity” the desired classification hinting that an asymptotic invariant is required; below we consider two examples of such invariants.

The *Jones polynomial* of the closure  $L$  of a braid  $b \in B_k$  is defined by the formula:

$$V_L(t) = \left( -\frac{t+1}{\sqrt{t}} \right)^{k-1} \text{tr} (r_t(b)), \quad (7)$$

where  $r_t$  is a representation of  $B_k$  in a von-Neumann algebra  $A_k$  and  $\text{tr}$  is a trace function; the  $V_L(t) \in \mathbf{Z}[t^{\pm \frac{1}{2}}]$ , i.e. a Laurent polynomial in the variable  $\sqrt{t}$  [Jones 1985] [10]. If  $K$  is the unknot then  $V_K(t) = 1$  and each polynomial  $V_L(t)$  can be calculated from  $K$  using the *skein relation*:

$$\frac{1}{t}V_{L^-} - tV_{L^+} = \left( \sqrt{t} - \frac{1}{\sqrt{t}} \right) V_L, \quad (8)$$

where  $L^+$  ( $L^-$ , resp.) is a link obtained by adding an overpass (underpass, resp.) to the link  $L$  [Jones 1985, Theorem 12] [10].

The *HOMFLY polynomial* of a link  $L$  is a Laurent polynomial  $\rho_L(l, m) \in \mathbf{Z}[l^{\pm 1}, m^{\pm 1}]$ ; the  $\rho_L(l, m)$  is defined recursively from the HOMFLY polynomial  $\rho_K(l, m) = 1$  of the unknot  $K$  using the skein relation:

$$l\rho_{L^+} + \frac{1}{l}\rho_{L^-} + m\rho_L = 0, \quad (9)$$

where  $L^+$  ( $L^-$ , resp.) is a link obtained by adding an overpass (underpass, resp.) to  $L$  [Freyd, Yetter, Hoste, Lickorish, Millet & Ocneanu 1985, Remark 3] [9].



### 3 Proof of theorem 1

For the sake of clarity, let us outline the main ideas. Roughly speaking, the Birman-Hilden's Theorem 2 says that a generator  $\sigma_i \in B_{2g+1}$  ( $B_{2g+2}$ , resp.) is given by the Dehn twist  $D_{\gamma_i} \in \text{Mod } S_{g,1}$  ( $\text{Mod } S_{g,2}$ , resp.) around a closed curve  $\gamma_i$ . The  $D_{\gamma_i}$  itself is a homeomorphism of the Teichmüller space  $T_{g,1}$  ( $T_{g,2}$ , resp.); therefore the  $D_{\gamma_i}$  induced an inner automorphism  $x \mapsto u_i x u_i^{-1}$ , where  $u_i$  is a unit of the cluster  $C^*$ -algebra  $\mathbb{A}(\mathbf{x}, S_{g,1})$  ( $\mathbb{A}(\mathbf{x}, S_{g,2})$ , resp.) (We refer the reader to Figure 1.) On the other hand, the Fomin-Shapiro-D. Thurston's Theorem 4 and remark 3 imply that units  $u_i$  and projections  $e_i$  in algebra  $\mathbb{A}(\mathbf{x}, S_{g,1})$  ( $\mathbb{A}(\mathbf{x}, S_{g,2})$ , resp.) are bijective. But the minimal degree polynomial in variable  $e_i$  corresponding to a unit  $u_i$  is a linear polynomial of the form  $u_i = ae_i + b$ , where  $a$  and  $b$  are complex constants. (The inverse is given by the formula  $u_i^{-1} = -\frac{a}{(a+b)b}e_i + \frac{1}{b}$ .) The Birman-Hilden's Theorem 2 implies that the  $u_i$  satisfy the braid relations  $\{u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}, u_i u_j = u_j u_i, \text{ if } |i - j| \geq 2\}$ ; moreover, if one substitutes  $u_i = ae_i + b$  in the braid relations, then:

$$\begin{cases} e_i^2 = e_i, \\ e_i e_{i\pm 1} e_i = -\frac{(a+b)b}{a^2} e_i, \\ e_i e_j = e_j e_i, \end{cases} \quad \text{if } |i - j| \geq 2. \quad (10)$$

**Remark 4** The relations (10) are invariant of the involution  $e_i^* = e_i$  if and only if  $\frac{(a+b)b}{a^2} \in \mathbf{R}$ ; in this case the algebra  $\mathbb{A}(\mathbf{x}, S_{g,1})$  ( $\mathbb{A}(\mathbf{x}, S_{g,2})$ , resp.) contains a finite-dimensional  $C^*$ -algebra  $\mathbb{A}_{2g}$  ( $\mathbb{A}_{2g+1}$ , resp.) obtained from the norm closure of a self-adjoint representation of a *Temperley-Lieb algebra*.

Thus one gets a representation  $\rho : B_{2g+1} \rightarrow \mathbb{A}(\mathbf{x}, S_{g,1})$  ( $B_{2g+2} \rightarrow \mathbb{A}(\mathbf{x}, S_{g,2})$ , resp.) given by the formula  $\sigma_i \mapsto ae_i + b$ , where  $1 \leq i \leq 2g$  ( $1 \leq i \leq 2g+1$ , resp.) It follows from relations (10) that the set  $\mathcal{E} := \{(e_{i_1} e_{i_1-1} \dots e_{j_1}) \dots (e_{i_p} e_{i_p-1} \dots e_{j_p}) \mid 1 \leq i_1 < \dots < i_p < 2g \text{ (} 2g+1, \text{ resp.)}; 1 \leq j_1 < \dots < j_p < 2g \text{ (} 2g+1, \text{ resp.)}; j_1 \leq i_1, \dots, j_p \leq i_p\}$  is multiplicatively closed; moreover,  $|\mathcal{E}| \leq \frac{1}{n+1} \binom{2n}{n} = n$ 'th Catalan number, where  $n = 2g$  ( $n = 2g+1$ , resp.) In particular, the  $\mathbb{A}_{2g}$  ( $\mathbb{A}_{2g+1}$ , resp.) of remark 4 is a finite-dimensional  $C^*$ -algebra and each element  $\varepsilon \in \mathcal{E}$  is equivalent to a projection; the (Murray-von Neumann) equivalence class of the projection will be denoted by  $[\varepsilon]$ . If  $\{b = \sigma_1^{k_1} \dots \sigma_{n-1}^{k_{n-1}} \in B_{n+1} \mid k_i \in \mathbf{Z}\}$  is a braid for  $n = 2g$

( $n = 2g + 1$ , resp.) then the polynomial  $\rho(b) = (ae_1 + b)^{k_1} \dots (ae_{n-1} + b)^{k_n}$  unfolds into a finite sum  $\left\{ \sum_{i=1}^{|\mathcal{E}|} a_i \varepsilon_i \mid \varepsilon_i \in \mathcal{E}, a_i \in \mathbf{Z} \right\}$ . Therefore one gets an inclusion  $[\rho(b)] := \left\{ \sum_{i=1}^{|\mathcal{E}|} a_i [\varepsilon_i] \mid [\varepsilon_i] \in K_0(\mathbb{A}(\mathbf{x}, S_{g,1})), a_i \in \mathbf{Z} \right\} \in K_0(\mathbb{A}(\mathbf{x}, S_{g,1}))$  ( $K_0(\mathbb{A}(\mathbf{x}, S_{g,2})$ , resp.) But  $K_0(\mathbb{A}(\mathbf{x}, S_{g,n})) \cong \mathcal{A}(\mathbf{x}, S_{g,n}) \subset \mathbf{Z}[\mathbf{x}^{\pm 1}]$ , where  $n = 1$  ( $n = 2$ , resp.); thus  $[\rho(b)] \in K_0(\mathbb{A}(\mathbf{x}, S_{g,n}))$  is a Laurent polynomial with the integer coefficients depending on  $2g$  ( $2g + 1$ , resp.) variables. The  $[\rho(b)]$  is in fact a topological invariant of the closure of  $b$ . Indeed,  $[\rho(gbg^{-1})] = [\rho(b)]$  for all  $b \in B_{2g+1}$  ( $B_{2g+2}$ , resp.) because the  $K_0$ -group and a canonical trace  $\tau$  on the algebra  $\mathbb{A}(\mathbf{x}, S_{g,1})$  ( $\mathbb{A}(\mathbf{x}, S_{g,2})$ , resp.) are related [Blackadar 1986, Section 7.3] [2]; since the  $\tau$  is a character of the representation  $\rho$ , one gets the formula  $[\rho(gbg^{-1})] = [\rho(b)]$ . An invariance of the  $[\rho(b)]$  with respect to the Markov move of type II is a bit subtler, but follows from a stability of the  $K$ -theory [Blackadar 1986, Section 5.1] [2]. Namely, the map  $\sigma_{2g+1}^{\pm 1} \mapsto 2e_{2g+1} - 1$  ( $\sigma_{2g+2}^{\pm 1} \mapsto 2e_{2g+2} - 1$ , resp.) gives rise to a crossed product  $C^*$ -algebra  $\mathbb{A}_{2g} \rtimes_{\alpha} G$  ( $\mathbb{A}_{2g+1} \rtimes_{\alpha} G$ , resp.), where  $G \cong \mathbf{Z}/2\mathbf{Z}$ ; the crossed product is isomorphic to the algebra  $M_2(\mathbb{A}_{2g}^{\alpha})$  ( $M_2(\mathbb{A}_{2g+1}^{\alpha})$ , resp.), where  $\mathbb{A}_{2g}^{\alpha}$  ( $\mathbb{A}_{2g+1}^{\alpha}$ , resp.) is the fixed-point algebra of the automorphism  $\alpha$  [Fillmore 1996, Section 3.8.5] [6]. But  $K_0(M_2(\mathbb{A}_{2g}^{\alpha})) \cong K_0(\mathbb{A}_{2g}^{\alpha})$  by the stability of the  $K$ -theory; the crossed product itself consists of the formal sums  $\sum_{\gamma \in G} a_{\gamma} u_{\gamma}$  and one easily derives that  $[\rho(b\sigma_{2g+1}^{\pm 1})] = [\rho(b)]$  for all  $b \in B_{2g+1}$  ( $B_{2g+2}$ , resp.)

We shall pass to a detailed proof of theorem 1 by splitting the argument into a series of lemmas.

**Lemma 1** *The map  $\{\sigma_i \mapsto e_i + 1 \mid 1 \leq i \leq 2g\}$  ( $\{\sigma_i \mapsto e_i + 1 \mid 1 \leq i \leq 2g + 1\}$ , resp.) defines a representation  $\rho : B_{2g+1} \rightarrow \mathbb{A}(\mathbf{x}, S_{g,1})$  ( $\rho : B_{2g+2} \rightarrow \mathbb{A}(\mathbf{x}, S_{g,2})$ , resp.) of the braid group with an odd (an even, resp.) number of strings into a cluster  $C^*$ -algebra of a Riemann surface with one cusps (two cusps, resp.)*

*Proof.* We shall prove the case  $\rho : B_{2g+1} \rightarrow \mathbb{A}(\mathbf{x}, S_{g,1})$  of the braid groups with an odd number of strings; the case of an even number of strings is treated likewise.

Let  $\{\gamma_1, \dots, \gamma_{2g}\}$  be a chain of simple closed curves on the surface  $S_{g,1}$ . The Dehn twists  $\{D_{\gamma_1}, \dots, D_{\gamma_{2g}}\}$  around  $\gamma_i$  satisfy the braid relations (3). The subgroup  $SMod S_{g,1}$  of  $Mod S_{g,1}$  consisting of the automorphisms of  $S_{g,1}$  commuting with the hyperelliptic involution  $\iota$  is isomorphic to the braid group  $B_{2g+1}$  (Birman-Hilden's Theorem 2).

On the other hand, each  $D_{\gamma_i}$  is a homeomorphism of the Teichmüller space  $T_{g,1}$ ; the  $D_{\gamma_i}$  induces an inner automorphism of the cluster  $C^*$ -algebra

$\mathbb{A}(\mathbf{x}, S_{g,1})$ , see remark 2. We shall denote such an automorphism by

$$\{u_i x u_i^{-1} \mid u_i \in \mathbb{A}(\mathbf{x}, S_{g,1}), \forall x \in \mathbb{A}(\mathbf{x}, S_{g,1})\}. \quad (11)$$

The units  $\{u_i \in \mathbb{A}(\mathbf{x}, S_{g,1}) \mid 1 \leq i \leq 2g\}$  satisfy the braid relations:

$$\begin{cases} u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}, \\ u_i u_j = u_j u_i, \end{cases} \quad \text{if } |i - j| \geq 2. \quad (12)$$

Indeed, from (3) one gets  $SMod S_{g,1} = \{D_{\gamma_1}, \dots, D_{\gamma_{2g}} \mid D_{\gamma_i} D_{\gamma_{i+1}} D_{\gamma_i} = D_{\gamma_{i+1}} D_{\gamma_i} D_{\gamma_{i+1}}, D_{\gamma_i} D_{\gamma_j} = D_{\gamma_j} D_{\gamma_i} \text{ if } |i - j| \geq 2\}$ . If  $Inn \mathbb{A}(\mathbf{x}, S_{g,1})$  is a group of the inner automorphisms of the algebra  $\mathbb{A}(\mathbf{x}, S_{g,1})$ , then  $\{u_i \mid 1 \leq i \leq 2g\}$  are generators of the  $Inn \mathbb{A}(\mathbf{x}, S_{g,1})$ . Using the commutative diagram in Figure 1, one gets from  $D_{\gamma_i} D_{\gamma_{i+1}} D_{\gamma_i} = D_{\gamma_{i+1}} D_{\gamma_i} D_{\gamma_{i+1}}$  the equality  $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ ; similarly, the  $D_{\gamma_i} D_{\gamma_j} = D_{\gamma_j} D_{\gamma_i}$  implies the equality  $u_i u_j = u_j u_i$  for  $|i - j| \geq 2$ . In particular,  $SMod S_{g,1} \cong Inn \mathbb{A}(\mathbf{x}, S_{g,1})$ , where the isomorphism is given by the formula  $D_{\gamma_i} \mapsto u_i$ .

It remains to express the units  $u_i$  in terms of generators  $e_i$  of the algebra  $\mathbb{A}(\mathbf{x}, S_{g,1})$ . The  $e_i$  itself is not invertible, but a polynomial  $u_i(e_i) = e_i + 1$  has an inverse  $u_i^{-1} = -\frac{1}{2}e_i + 1$ . From an isomorphism  $B_{2g+1} \cong SMod S_{g,1}$  given by the formula  $\sigma_i \mapsto D_{\gamma_i}$  one gets a representation  $\rho : B_{2g+1} \rightarrow \mathbb{A}(\mathbf{x}, S_{g,1})$  given by the formula  $\sigma_i \mapsto e_i + 1$ . Lemma 1 is proved.

**Corollary 1** *The norm closure of a self-adjoint representation of a Temperley-Lieb algebra  $TL_{2g}(\frac{i\sqrt{2}}{2})$  ( $TL_{2g+1}(\frac{i\sqrt{2}}{2})$ , resp.) is a finite-dimensional sub- $C^*$ -algebra  $\mathbb{A}_{2g}$  ( $\mathbb{A}_{2g+1}$ , resp.) of the  $\mathbb{A}(\mathbf{x}, S_{g,1})$  ( $\mathbb{A}(\mathbf{x}, S_{g,2})$ , resp.)*

*Proof.* Let us substitute  $u_i = e_i + 1$  into the braid relations (12). The reader is encouraged to verify that relations (12) are equivalent to the following system of relations:

$$\begin{cases} e_i^2 = e_i, \quad e_i^* = e_i, \\ e_i e_{i\pm 1} e_i = -2e_i, \\ e_i e_j = e_j e_i, \end{cases} \quad \text{if } |i - j| \geq 2. \quad (13)$$

A normalization  $e'_i = \frac{i\sqrt{2}}{2}e_i$  brings (13) to the form:

$$\begin{cases} e_i^2 = \frac{i\sqrt{2}}{2}e_i, \\ e_i e_{i\pm 1} e_i = e_i, \\ e_i e_j = e_j e_i, \end{cases} \quad \text{if } |i - j| \geq 2. \quad (14)$$

The relations (14) are defining relations for a Temperley-Lieb algebra  $TL_{2g}(\frac{i\sqrt{2}}{2})$  ( $TL_{2g+1}(\frac{i\sqrt{2}}{2})$ , resp.), see e.g. [Jones 1991, p. 85] [11]; such an algebra is always finite-dimensional, see next lemma. Corollary 1 follows.

**Lemma 2** ([Jones 1991, Section 3.5] [11]) *The set  $\mathcal{E} := \{(e_{i_1}e_{i_1-1}\dots e_{j_1})\dots(e_{i_p}e_{i_p-1}\dots e_{j_p}) \mid 1 \leq i_1 < \dots < i_p < 2g \text{ (} 2g+1, \text{ resp.)}; 1 \leq j_1 < \dots < j_p < 2g \text{ (} 2g+1, \text{ resp.)}; j_1 \leq i_1, \dots, j_p \leq i_p\}$  is multiplicatively closed and*

$$|\mathcal{E}| \leq \frac{1}{n+1} \binom{2n}{n}, \quad (15)$$

where  $n = 2g$  ( $n = 2g+1$ , resp.)

*Proof.* An elegant proof of this fact is based on a representation of the relations (14) by the diagrams of the non-crossing strings reminiscent of the braid diagrams.

**Corollary 2** *Each element  $e \in \mathcal{E}$  is equivalent to a projection in the cluster  $C^*$ -algebra  $\mathbb{A}(\mathbf{x}, S_{g,1})$  ( $\mathbb{A}(\mathbf{x}, S_{g,2})$ , resp.)*

*Proof.* Indeed, if  $e \in \mathcal{E}$  then the  $e^2$  must coincide with one of the elements of  $\mathcal{E}$ . But  $e^2$  cannot be any such, except for the  $e$  itself. Thus  $e^2 = e$ , i.e.  $e$  is an idempotent. On the other hand, it is well known that each idempotent in a  $C^*$ -algebra is (Murray-von Neumann) equivalent to a projection in the same algebra, see e.g. [Blackadar 1986, Proposition 4.6.2] [2]. Corollary 2 follows.

**Lemma 3** *If  $b \in B_{2g+1}$  ( $b \in B_{2g+2}$ , resp.) is a braid, there exists a Laurent polynomial  $[\rho(b)]$  with the integer coefficients depending on  $2g$  ( $2g+1$ , resp.) variables, such that  $[\rho(b)] \in K_0(\mathbb{A}(\mathbf{x}, S_{g,1}))$  ( $[\rho(b)] \in K_0(\mathbb{A}(\mathbf{x}, S_{g,2}))$ , resp.)*

*Proof.* We shall prove this fact for the braid groups with an odd number of strings; the case of an even number of strings is treated likewise.

Let  $\{b = \sigma_1^{k_1} \dots \sigma_{2g}^{k_{2g}} \in B_{2g+1} \mid k_i \in \mathbf{Z}\}$  be a braid. By lemma 1 such a braid has a representation  $\rho(b)$  in the cluster  $C^*$ -algebra  $\mathbb{A}(\mathbf{x}, S_{g,1})$  given by the formula:

$$\rho(b) = (e_1 + 1)^{k_1} \dots (e_{2g} + 1)^{k_{2g}} \in \mathbb{A}(\mathbf{x}, S_{g,1}). \quad (16)$$

One can unfold the product (16) into a sum of the monomials in variables  $e_i$ ; by lemma 2 any such monomial is an element of the set  $\mathcal{E}$ . In other words, one gets a finite sum:

$$\rho(b) = \left\{ \sum_{i=1}^{|\mathcal{E}|} a_i \varepsilon_i \mid \varepsilon_i \in \mathcal{E}, a_i \in \mathbf{Z} \right\}. \quad (17)$$

(Note that whenever  $k_i < 0$  the coefficient  $a_i$  is a rational number, but clearing the denominators we can assume  $a_i \in \mathbf{Z}$ .) On the other hand, corollary 2 says that each  $\varepsilon_i$  is a projection; therefore  $\varepsilon_i$  defines an equivalence class  $[\varepsilon_i] \in K_0(\mathbb{A}(\mathbf{x}, S_{g,1}))$  of projections in the cluster  $C^*$ -algebra  $\mathbb{A}(\mathbf{x}, S_{g,1})$ . Thus one gets  $[\rho(b)] \in K_0(\mathbb{A}(\mathbf{x}, S_{g,1}))$  given by a finite sum

$$[\rho(b)] = \left\{ \sum_{i=1}^{|\mathcal{E}|} a_i [\varepsilon_i] \mid [\varepsilon_i] \in K_0(\mathbb{A}(\mathbf{x}, S_{g,1})), a_i \in \mathbf{Z} \right\}. \quad (18)$$

But  $K_0(\mathbb{A}(\mathbf{x}, S_{g,1})) \cong \mathcal{A}(\mathbf{x}, S_{g,1}) \subset \mathbf{Z}[\mathbf{x}^{\pm 1}]$ ; in particular,  $[\rho(b)] \in \mathbf{Z}[\mathbf{x}^{\pm 1}]$  is a Laurent polynomial with the integer coefficients.

To calculate the number of variables in  $[\rho(b)]$ , recall that  $\text{rank } \mathcal{A}(\mathbf{x}, S_{g,1}) = 6g - 3$ ; on the other hand, a fundamental domain of the Riemann surface  $S_{g,1}$  is a paired  $(4g + 2)$ -gon, where one pair of sides corresponds to a boundary component obtained from the cusp, see Section 2.1. Since the boundary component contracts to a cusp, one gets a paired  $4g$ -gon whose triangulation requires  $4g - 3$  interior geodesic arcs. Thus the cluster  $|\mathbf{x}| = 6g - 3$  can be written in the form:

$$\mathbf{x} = (x_1, \dots, x_{2g}; y_1, \dots, y_{4g-3}), \quad (19)$$

where  $x_i$  are mutable and  $y_i$  are frozen variables [Williams 2014, Definition 2.6] [16]. One can always assume  $y_i = \text{Const}$  and therefore the Laurent polynomial  $[\rho(b)]$  depends on the  $2g$  variables  $x_i$ . Lemma 3 follows.

**Lemma 4** *The Laurent polynomial  $[\rho(b)]$  is a topological invariant of the closure of the braid  $b \in B_{2g+1}$  ( $b \in B_{2g+2}$ , resp.)*

*Proof.* Again we shall prove the case  $b \in B_{2g+1}$ ; the case  $b \in B_{2g+2}$  can be treated similarly and is left to the reader. To prove that  $[\rho(b)]$  is a topological invariant, it is enough to demonstrate that:

- (i)  $[\rho(gbg^{-1})] = [\rho(b)]$  for all  $g \in B_{2g+1}$ ;
- (ii)  $[\rho(b\sigma_{2g+1}^{\pm 1})] = [\rho(b)]$  for the generator  $\sigma_{2g+1} \in B_{2g+2}$ .

(i) Recall that the  $K$ -theory of an  $AF$ -algebra  $\mathbb{A}(\mathbf{x}, S_{g,1})$  can be recovered from the canonical trace  $\tau : \mathbb{A}(\mathbf{x}, S_{g,1}) \rightarrow \mathbb{C}$  [Blackadar 1986, Section 7.3] [2]. On the other hand, the trace  $\tau$  is a character of the representation  $\rho : B_{2g+1} \rightarrow \mathbb{A}(\mathbf{x}, S_{g,1})$ ; in particular,  $\tau(\rho(g)\rho(b)\rho(g^{-1})) = \tau(\rho(b))$ . In other words,  $[\rho(gbg^{-1})] = [\rho(b)]$  for all  $g \in B_{2g+1}$ . Item (i) follows.

(ii) Let  $\mathbb{A}_{2g}$  be a finite-dimensional  $C^*$ -algebra of corollary 1. Let  $u_{2g+1}$  and  $u_{2g+1}^{-1}$  be a unit and its inverse in the algebra  $\mathbb{A}(\mathbf{x}, S_{g,1})$  given by the formulas:

$$u_{2g+1} = u_{2g+1}^{-1} := 2e_{2g+1} - 1. \quad (20)$$

Clearly the units  $u_{2g+1}^{\pm 1} \notin \mathbb{A}_{2g}$  and they make the group  $G \cong \mathbb{Z}/2\mathbb{Z}$  under a multiplication. We shall consider an extension  $\mathbb{A}_{2g} \rtimes G$  of the algebra  $\mathbb{A}_{2g}$  given by the formal sums

$$\mathbb{A}_{2g} \rtimes G := \left\{ \sum_{\gamma \in G} a_{\gamma} u_{\gamma} \mid u_{\gamma} \in \mathbb{A}_{2g} \right\}. \quad (21)$$

(The algebra of formal sums (21) is isomorphic to a crossed product  $C^*$ -algebra of the algebra  $\mathbb{A}_{2g}$  by the outer automorphisms  $\alpha$  given the elements  $u_{\gamma} \in \{u_{2g+1}^{\pm 1}\}$ ; hence our notation.) It is well known that

$$\mathbb{A}_{2g} \rtimes G \cong M_2(\mathbb{A}_{2g}^{\alpha}), \quad (22)$$

where  $\mathbb{A}_{2g}^{\alpha} \subset \mathbb{A}_{2g}$  is a fixed-point algebra of the automorphism  $\alpha : \mathbb{A}_{2g} \rightarrow \mathbb{A}_{2g}$ , see e.g. [Fillmore 1996, Section 3.8.5] [6]. By the second of formulas (12), one gets  $u_i u_{2g+1} = u_{2g+1} u_i$  for all  $1 \leq i \leq 2g-1$ ; in other words,

$$u_i = u_{2g+1} u_i u_{2g+1}^{-1} \quad (23)$$

for all  $1 \leq i \leq 2g-1$ . Since all generators  $u_i$  of algebra  $\mathbb{A}_{2g-1}$  are fixed by the automorphism  $\alpha$ , one gets an isomorphism

$$\mathbb{A}_{2g}^{\alpha} \cong \mathbb{A}_{2g-1}. \quad (24)$$

On the other hand,  $K_0(M_2(A)) \cong K_0(A)$  by a stability of the  $K$ -theory [Blackadar 1986, Section 5.1] [2]; thus formulas (22) and (24) imply an isomorphism:

$$K_0(\mathbb{A}_{2g} \rtimes G) \cong K_0(\mathbb{A}_{2g-1}). \quad (25)$$

It remains to notice that if one maps the generators  $\sigma_{2g+1}^{\pm 1} \in B_{2g+2}$  into the units  $u_{2g+1}^{\pm 1} \in \mathbb{A}_{2g} \rtimes G$ , then formulas (21) and (25) imply the equality

$$[\rho(b\sigma_{2g+1}^{\pm 1})] = [\rho(b)] \quad (26)$$

for all  $b \in B_{2g}$ . The item (ii) follows from (26) and lemma 4 is proved.

Theorem 1 follows from lemmas 1, 3 and 4.

## 4 Examples

To illustrate theorem 1, we shall consider a representation  $\rho : B_2 \rightarrow \mathbb{A}(\mathbf{x}, S_{0,2})$  ( $\rho : B_3 \rightarrow \mathbb{A}(\mathbf{x}, S_{1,1})$ , resp.); it will be shown that for such a representation the Laurent polynomials  $[\rho(b)]$  correspond to the Jones (HOMFLY, resp.) invariants of knots and links.

### 4.1 Jones polynomials

If  $g = 0$  and  $n = 2$ , then  $S_{0,2}$  is a sphere with two cusps; the  $S_{0,2}$  is homotopy equivalent to an annulus  $\mathfrak{A} := \{z = u + iv \in \mathbf{C} \mid r \leq |z| \leq R\}$ . The Riemann surface  $\mathfrak{A}$  has an ideal triangulation  $T$  with one marked point on each boundary component given by the matrix:

$$B_T = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \quad (27)$$

see [Fomin, Shapiro & Thurston 2008, Example 4.4] [8]. Using the exchange relations (1) the reader can verify that the cluster  $C^*$ -algebra  $\mathbb{A}(\mathbf{x}, S_{0,2})$  is given by the Bratteli diagram shown in Figure 2; the  $\mathbb{A}(\mathbf{x}, S_{0,2})$  coincides with the so-called *GICAR algebra* [Bratteli 1972, Section 5.5] [3].

The cluster  $\mathbf{x} = (x; c)$  consists of a mutable variable  $x$  and a coefficient  $c \in (\mathbb{P}, \oplus, \bullet)$ . Theorem 1 says that there exists a representation

$$\rho : B_2 \rightarrow \mathbb{A}(\mathbf{x}, S_{0,2}), \quad (28)$$

such that  $[\rho(b)] \in \mathbf{Z}[x^{\pm 1}]$  is a topological invariant of the closure  $L$  of  $b \in B_2$ ; since the Laurent polynomial  $[\rho(b)]$  depends on  $x$  and  $c$ , we shall write it  $[\rho(b)](x, c)$ . Let  $N \geq 1$  be the minimal number of the overpass (underpass, resp.) crossings added to the unknot  $K$  to get the link  $L$  [Jones 1985, Figure 2] [10]. The following result compares the  $[\rho(b)](x, c)$  with the Jones polynomial  $V_L(t)$ .

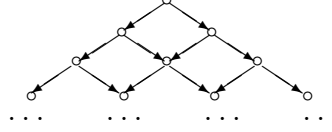


Figure 2: Bratteli diagram of the algebra  $\mathbb{A}(\mathbf{x}, S_{0,2})$ .

**Corollary 3**

$$V_L(t) = \left( -\frac{\sqrt{t}}{t+1} \right)^N [\rho(b)](t, -t^2). \quad (29)$$

*Proof.* Recall that each polynomial  $[\rho(b)](x, c)$  is obtained from an initial seed  $(\mathbf{x}, B_T)$  by a finite number of mutation given by the exchange relations (1); likewise, each polynomial  $V_L(t)$  can be obtained from the  $V_K(t) = 1$  using the skein relation (8). Roughly speaking, the idea is to show that (1) and (8) are equivalent relations up to a multiple  $-\frac{\sqrt{t}}{t+1}$ . Indeed, consider the Laurent polynomials:

$$\begin{cases} W_{L^+} = \left( -\frac{t+1}{\sqrt{t}} \right) V_{L^+} \\ W_{L^-} = \left( -\frac{t+1}{\sqrt{t}} \right) V_{L^-}. \end{cases} \quad (30)$$

The skein relation (8) for the  $W_{L^\pm}$  takes the form:

$$V_L = \frac{t^2}{t^2 - 1} W_{L^+} - \frac{1}{t^2 - 1} W_{L^-}. \quad (31)$$

The substitution

$$\begin{cases} V_L = x'_k \\ W_{L^+} = \frac{1}{x_k} \prod_{i=1}^n x_i^{\max(b_{ik}, 0)} \\ W_{L^-} = \frac{1}{x_k} \prod_{i=1}^n x_i^{\max(-b_{ik}, 0)} \\ c_k = -t^2 \end{cases} \quad (32)$$

transforms the skein relation (31) to the exchange relations (1). It remains to observe from (30), that an extra crossing added to the unknot  $K$  corresponds to a multiplication of  $[\rho(b)](x, c)$  by  $-\frac{\sqrt{t}}{t+1}$ ; the minimal number  $N$  of such crossings required to get the link  $L$  from  $K$  gives us the  $N$ -th power of the multiple. Corollary 3 follows.



## 4.2 HOMFLY polynomials

If  $g = n = 1$ , then  $S_{1,1}$  is a torus with a cusp. The matrix  $B_T$  associated to an ideal triangulation of the Riemann surface  $S_{1,1}$  has the form:

$$B_T = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}, \quad (33)$$

see [Fomin, Shapiro & Thurston 2008, Example 4.6] [8]. Using the exchange relations (1) the reader can verify that the cluster  $C^*$ -algebra  $\mathbb{A}(\mathbf{x}, S_{1,1})$  is given by the Bratteli diagram shown in Figure 3; the  $\mathbb{A}(\mathbf{x}, S_{1,1})$  coincides with the *Mundici algebra*  $\mathfrak{M}_1$  [Mundici 1988] [12].

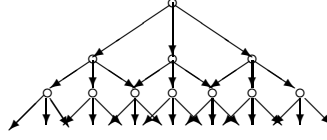


Figure 3: Bratteli diagram of the algebra  $\mathbb{A}(\mathbf{x}, S_{1,1})$ .

The formula (19) implies that cluster  $\mathbf{x} = (x_1, x_2; y_1)$  consists of two mutable variables  $x_1, x_2$  and a frozen variable  $y_1$ . Theorem 1 says that there exists a representation:

$$\rho : B_3 \rightarrow \mathbb{A}(\mathbf{x}, S_{1,1}), \quad (34)$$

such that  $[\rho(b)] \in \mathbf{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$  is a topological invariant of the closure  $L$  of  $b \in B_3$ ; since the Laurent polynomial  $[\rho(b)]$  depends on two variables  $x_1, x_2$  and two coefficients  $c_1, c_2 \in (\mathbb{P}, \oplus, \bullet)$ , we shall write it  $[\rho(b)](x_1, x_2; c_1, c_2)$ . The following result says that the  $[\rho(b)](x_1, x_2; c_1, c_2)$  for special values of  $c_i$  is related to the HOMFLY polynomial  $\rho_L(l, m)$ . (Unlike (29), an explicit formula for such a relationship seems to be complicated.)

**Corollary 4** *The exchange relations (1) with matrix  $B$  given by (33) imply the skein relation (9) for the HOMFLY polynomial  $\rho_L(l, m)$ .*

*Proof.* Since the variable  $y_1$  is frozen, we consider a reduced matrix:

$$\tilde{B}_T = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \quad (35)$$

and our seed has the form  $(\mathbf{x}, \tilde{B}_T)$ , where  $\mathbf{x} = (x_1, x_2; c_1, c_2)$ . The exchange relations (1) for the variables  $x_3, x_4, x_5$  and the coefficient  $c_3$  imply the following system of equations:

$$\begin{cases} x_3 = \frac{c_1 + x_2^2}{(c_1 + 1)x_1} \\ x_4 = \frac{c_2 x_3^2 + 1}{(c_2 + 1)x_2} \\ x_5 = \frac{c_3 + x_4^2}{(c_3 + 1)x_3} \\ c_3 = \frac{1}{c_1}. \end{cases} \quad (36)$$

Clearing the denominators in (36), one gets an expression:

$$c_1 x_3 + c_3 x_5 = \frac{c_1 + x_2^2 - x_1 x_3}{x_1} + \frac{c_3 + x_4^2 - x_3 x_5}{x_3}. \quad (37)$$

We exclude  $x_4 = \frac{c_2 x_3^2 + 1}{(c_2 + 1)x_2}$  and  $c_3 = \frac{1}{c_1}$  in (37) and get the following equality of the Laurent polynomials:

$$c_1 x_3 + \frac{1}{c_1} x_5 + c_2 W = 0, \quad (38)$$

where  $W := -c_2 \left[ \frac{c_1 + x_2^2 - x_1 x_3}{c_2^2 x_1} + \frac{c_1^{-1} - x_3 x_5}{c_2^2 x_3} + \frac{1}{x_3} \left( \frac{c_2 x_3^2 + 1}{c_2(c_2 + 1)x_2} \right)^2 \right]$ . The substitution:

$$\begin{cases} c_1 = x_1 = l \\ c_2 = x_2 = m \\ x_3 = \rho_{L^+} \\ x_5 = \rho_{L^-} \\ W = \rho_L \end{cases} \quad (39)$$

brings (38) to the skein relation (9). Corollary 4 follows.

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THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES,  
TORONTO, ON, CANADA, E-MAIL: [igor.v.nikolaev@gmail.com](mailto:igor.v.nikolaev@gmail.com)